Single Pure - Chain, Product And Quotient Rules

1. Differentiate the following and fully simplify your answer.

   (a) $y = x^4(2x-1)^5$. $\boxed{\frac{dy}{dx} = 2x^3(9x-2)(2x-1)^4}$

   (b) $y = 3x^7(3x^2+4)^5$. $\boxed{\frac{dy}{dx} = 3x^6(51x^2+28)(3x^2+4)^4}$

   (c) $y = 2x^5(3-x)^3$. $\boxed{\frac{dy}{dx} = 2x^4(15-8x)(3-x)^2}$

   (d) $y = (x-1)^9(x+1)^3$. $\boxed{\frac{dy}{dx} = 6(2x+1)(x+1)^2(x-1)^8}$

   (e) $y = (2-x)^9(2+x)^9$. $\boxed{\frac{dy}{dx} = -18x(2-x)^8(2+x)^8}$

   (f) $y = 2(2x+1)^8(3x-1)^4$. $\boxed{\frac{dy}{dx} = 8(18x-1)(3x-1)^3(2x+1)^7}$

   (g) $y = \sqrt{2x-1}(x+3)^8$. $\boxed{\frac{dy}{dx} = \frac{(17x-5)(x+3)^7}{\sqrt{2x-1}}}$

   (h) $y = \sqrt{1-4x}(2x-1)^4$. $\boxed{\frac{dy}{dx} = \frac{2(2x-1)^3(5-18x)}{\sqrt{1-4x}}}$

   (i) $y = \dfrac{x^3}{(x-1)^8}$. $\boxed{\frac{dy}{dx} = -\frac{x^2(5x+3)}{(x-1)^9}}$

   (j) $y = \dfrac{(2x-1)^5}{(4x+1)^{10}}$. $\boxed{\frac{dy}{dx} = \frac{10(5-4x)(2x-1)^4}{(4x+1)^{11}}}$

   (k) $y = \dfrac{(3-2x)^4}{\sqrt{8x+3}}$. $\boxed{\frac{dy}{dx} = -\frac{4(14x+9)(3-2x)^3}{(8x+3)^{\frac{3}{2}}}}$

   (l) $y = \dfrac{(2x+1)^{10}}{(1+2x)^9}$. $\boxed{\frac{dy}{dx} = 2}$

2. Find the x-coordinates of the stationary points on the curve $y = ax^n(bx+c)^m$. $\boxed{x = 0, x = -\frac{c}{b}, x = -\frac{cn}{b(m+n)}}$

Quotient Rule

Differentiate $y = \frac{x^2+1}{\sqrt{8x-1}}$.

Preliminaries

You will need the results

$$y = \ln x \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{1}{x}$$

and

$$y = e^x \qquad \Rightarrow \qquad \frac{dy}{dx} = e^x.$$

The Chain Rule

The chain rule states that

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

What this means is that if we can see a function "wrapped up" in another, then we can use the result of the chain rule to differentiate it. For example in the case of $y = (x^2 + 1)^{10}$ we let $u = x^2 + 1$. We need dy/du and du/dx so we set out as follows:

$$y = u^{10} \qquad u = x^2 + 1$$

$$\frac{dy}{du} = 10u^9 \qquad \frac{du}{dx} = 2x$$

and by chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 10u^9 \times 2x = 20x(x^2 + 1)^9.$$

Notice we have introduced u to help us through, but then not used it at all in our final answer. I say we have "used and abused" u to help us through the question. The results above can be generalised to give

$$y = [f(x)]^n \text{ then } \frac{dy}{dx} = nf'(x)[f(x)]^{n-1}.$$

In another example, involving e, we may wish to differentiate $3e^{x^3+x}$. As before let the 'complicated bit' be u and proceed:

$$y = 3e^u \qquad u = x^3 + x$$

$$\frac{dy}{du} = 3e^u \qquad \frac{du}{dx} = 3x^2 + 1$$

and by chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3e^u \times (3x^2 + 1) = 3(3x^2 + 1)e^{x^3+x}.$$

The Product Rule

The product rule allows us to differentiate a function when it is comprised of a product of two functions; i.e. $y = uv = u(x) \times v(x)$. It states that

$$y = uv \qquad \Rightarrow \qquad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

In other words you break up your function into two bits (your u and v) and then differentiate one and leave the other alone and add in the result when you do it the other way round. For example with $y = 3x^2(3x+1)^5$ I would let $u = 3x^2$ and $v = (3x+1)^5$. We know that $\frac{du}{dx} = 6x$ and by the chain rule above we find that $\frac{dv}{dx} = 15(3x+1)^4$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= 3x^2 \times 15(3x+1)^4 + 6x(3x+1)^5\\ &= 45x^2(3x+1)^4 + 6x(3x+1)^5\\ &= 3x(3x+1)^4\left[15x + 2(3x+1)\right]\\ &= 3x(3x+1)^4(21x+2).\end{aligned}$$

Notice how I have completely factorised the answer; you must do this when you use the product rule. The factorised form at the end is by far the most useful form.

Questions

Combining these ideas differentiate and simplify the following:

1. $y = 2(x^2+x)^7$. $\boxed{14(2x+1)(x^2+x)^6}$
2. $y = e^{3x+1}$. $\boxed{3e^{3x+1}}$
3. x^3e^x. $\boxed{x^2e^x(x+3)}$
4. $y = \ln(3x^2+1)$. $\boxed{\frac{6x}{3x^2+1}}$
5. $y = 2x(1+x)^8$. $\boxed{2(1+9x)(1+x)^7}$
6. $y = 2x^2(x^2+1)^4$. $\boxed{4x(5x^2+1)(x^2+1)^3}$
7. $y = \dfrac{x^3}{x^2+1}$. $\boxed{\frac{x^2(x^2+3)}{(x^2+1)^2}}$
8. $y = x^2e^{2x}$. $\boxed{2x(1+x)e^{2x}}$
9. $y = 3x^2\sqrt{1-x^2}$. $\boxed{\frac{3x(2-3x^2)}{\sqrt{1-x^2}}}$
10. $y = \frac{\sqrt[3]{3x+1}}{x-1}$. $\boxed{-\frac{2(x+1)}{(x-1)^2(3x+1)^{\frac{2}{3}}}}$
11. $y = \dfrac{x^2+1}{1-x}$. $\boxed{\frac{1+2x-x^2}{(1-x)^2}}$
12. $y = \dfrac{1+x^2}{\sqrt{x-1}}$. $\boxed{\frac{3x^2-4x-1}{2(x-1)^{3/2}}}$
13. $y = e^x\sqrt{5x^2+2}$. $\boxed{\frac{e^x(5x^2+5x+2)}{\sqrt{5x^2+2}}}$
14. $y = \frac{x^2+1}{\sqrt{x^2+2}}$. $\boxed{\frac{x(x^2+3)}{(x^2+2)^{\frac{3}{2}}}}$

Chain Rule

The chain rule states that

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

You can always do it the really slow way that the textbooks give, but you should be looking to do this *fast*. Try to do the following as quick as you can, making no mistakes.

Remember the following:

y	$\frac{dy}{dx}$
ax^n	anx^{n-1}
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$

1. Differentiate the following functions (with respect to x) by means of the chain rule:

    (a) $y = (x+3)^5$.
    (b) $y = (2x-5)^8$.
    (c) $y = (x^3+2)^6$.
    (d) $y = (2x-1)^{-2}$.
    (e) $y = \sqrt{x^2+1}$.
    (f) $y = \frac{1}{(x^2-3)^3}$.
    (g) $y = (\sqrt{x}-1)^4$.
    (h) $y = (\frac{1}{x}-1)^n$.
    (i) $y = (ax^2+bx+c)^n$.
    (j) $y = e^{2x}$.
    (k) $y = e^{-5x}$.
    (l) $y = 4e^{x^2}$.
    (m) $y = e^{x^2-2x+1}$.
    (n) $y = \sin(x^2)$.
    (o) $y = \cos(x^3+4)$.
    (p) $y = \sin(\sin(x^2))$.
    (q) $y = \sin(\cos e^{x^5})$.
    (r) $y = \ln 3x$.
    (s) $y = \ln(x^2+4x-3)$.
    (t) $y = \ln \sin x$.
    (u) $y = \ln \cos e^{\sin x}$.
    (v) $y = e^{(e^{3x})}$.
    (w) $y = \sin\sin\sin\sin\sin\sin(x^3)$

Product Rule

The product rules states that if $y = uv$ where u and v are functions of x, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Remember that to factorise things you **pull out the lowest powers**! For example:

$$x^2(2x+1)^{-4} + x^3(2x+1)^{-5} = x^2(2x+1)^{-5}\big[(2x+1)+x\big] = x^2(2x+1)^{-5}(3x+1).$$

1. Differentiate the following functions (with respect to x) by means of the chain rule:

   (a) $y = x^3(x+1)^{10}$.